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A CLASS OF HIGHER FRACTIONAL DIFFERENTIAL EQUATION AND OPTIMAL CONTROL

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Abstract. In this paper the existence and uniqueness of mild solution of Hilfer fractional differential system with Poisson jump and fractional Brownian motion are discussed using stochastic analysis,compact semigroup theory and fractional calculus. Also, the existence of optimal control pair for the Hilfer fractional control system is discussed by taking general mild assumptions of cost functional.

Keywords : Optimal Control ; Hilfer Fractional differential equation ; Fractional Brownian motion; Nonlocal condition

AMS (MOS) Subject Classification : 45J05, 34K30, 34K40, 65L03, 34G20, 34A37, 93B05

1. Introduction. The definition of Hilfer fractional derivative connects the definitions of Caputo and Riemann-Liouville fractional derivatives. The two parameters of Hilfer fractional derivative allows an extra degree of freedom on initial conditions. Also, this gives rise more types of stationary states. It appears, for example, in theoretical simulation of dielectric relaxation in glass forming materials. For more details one may refer (Li, 2015, Liu, Yan, Cang Y., 2012, Mishra, 2008, and oaraless, (Dedgeade, and Li, 2015, Liu, Yan, Gang, 2012 and Morales-Delgado, and Morales-Delgado, Gmez-Aguilar, Saad, Khan and Agarwal, 2019) .

Brownian motion alone fails to depict sudden fluctuations in some real systems such as in financial instruments with jumps. Derivatives of stocks, such as Vanilla options, etc., are some examples of financial instruments. Therefore, in this paper we incorporated Poisson jumps to illustrate such stochastic systems. There are some instances such as present stock price might influence the price in future. So, in such scenarios Brownian motion which has independent increment property has no use. Rather fractional Brownian motion is more suitable for these occasions.

Very often, optimal control problems arise in system engineering. Main aim is to find a control that optimizes a cost functional or a performance index. The task of finding such control is open ended because of the complexity of fractional stochastic nonlinear systems. Therefore, in this paper the existence and uniqueness of mild solution of the following Hilfer fractional differential system with Poisson jump and fractional Brownian motion have been discussed using stochastic analysis, compact semigroup theory and fractional analysis.

$$\begin{aligned} D^{\upsilon,\mu}x(t) &= Ax(t) + f(t,x(t)) + G(t,x(t))\frac{dB^H(t)}{dt} \\ &\quad + \int_Y p(t,x(t),y)\widehat{N}(dt,dy), \quad t \in (0,b] \\ I_0^{(1-\upsilon)(1-\mu)}x(0) + g(x) &= x_0 \end{aligned} \tag{1.1}$$

Also the existence of optimal control pairs for the Hilfer fractional control system is discussed by taking general mild assumptions of cost functional.

2. Preliminaries. The state variable takes values in Hilbert space $\mathcal{H}$. Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space with $\{B^H(t)\}_{t\geq 0}$ denoting a fractional Brownian motion (fBm) with Hurst parameter $H \in (1/2, 1)$ defined on $(\Omega, \mathfrak{F}, P)$ with values in Hilbert space Y. $\widehat{N}(ds,dy) = N(ds,dy) - \lambda(dy)ds$ and $N(ds,dy)$ denotes compensating martingale measure and Poisson counting measure respectively. Here λ denotes the conditional jump intensity or equivalently, the average number of jumps per unit time. In the language of finance, an equilibrium measure or a risk-neutral measure, or equivalently a martingale measure, is defined as the probability measure under which every share price matches the discounted expectation of the share price itself.

$C^{\upsilon,\mu}(J, L_2(\Omega,\mathcal{H})) = \{x \in C((0,b], L_2(\Omega,\mathcal{H})); \lim_{t\to 0^+} t^{(1-\upsilon)(1-\mu)}x(t)$ exists and its finite$\}$ with norm defined as

$$\|x\|^2_{\upsilon,\mu} = \sup_{0<t\leq b} \|t^{(1-\upsilon)(1-\mu)}x(t)\|^2$$

is clearly a Banach space. Here A is the infinitesimal generator of a strongly continuous semigroup $T(t)_{t\geq 0}$.

DEFINITION 1. *Fractional integral of order $\alpha > 0$ with lower limit a is defined as*

$$I^\alpha_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(s)}{(t-s)^{1-\alpha}}ds$$

DEFINITION 2. *Hilfer fractional derivative of order* $0 \leq \upsilon \leq 1$ *and* $0 < \mu < 1$ *with lower limit a is defined as*

$$D^{\upsilon,\mu} f(t) = I_{a^+}^{\upsilon(1-\mu)} \frac{d}{dt} I_{a^+}^{(1-\upsilon)(1-\mu)} f(t)$$

The operator $T_\mu(t) = \int_0^\infty \mu\phi\psi_\mu(\phi) T(t^\mu \phi) d\phi$, where

$$\psi_\mu(\phi) = \sum_{n=1}^{\infty} ((-\phi)^{n-1}/(n-1)!\Gamma(1-n\mu)) \sin(n\pi\alpha), \phi \in (0,\infty)$$

$$\textit{and} \quad P_\mu(t) = t^{\mu-1} T_\mu(t).$$

REMARK. $D^{\upsilon(1-\mu)} S_{\upsilon,\mu}(t) = P_\mu(t)$

DEFINITION 3. *The mild solution of* (1.1) *is defined as*

$$\begin{aligned} x(t) = S_{\upsilon,\mu}(t)(x_0 - g(x)) &+ \int_0^t P_\mu(t-s) f(s, x(s)) ds \\ &+ \int_0^t P_\mu(t-s) G(s, x(s)) dB^H(s) \\ &+ \int_0^t P_\mu(t-s) \int_Y p(s, x(s), y) \widehat{N}(ds, dy), \quad \forall t \in (0, b] \end{aligned} \tag{2.1}$$

The following assumptions are made

H1: The functions $f : J \times \mathcal{H} \to H$, $G : J \times \mathcal{H} \to L_2^0(Y, \mathcal{H})$, $g : C(J, \mathcal{H}) \to \mathcal{H}$, $p : J \times \mathcal{H} \times Y \to \mathcal{H}$ satisfy

$$\|f(t, x_1) - f(t, x_2)\|^2 \leq K(\|x_1 - x_2\|^2),$$
$$\|g(x_1) - g(x_2)\|^2 \leq K(\|x_1 - x_2\|^2),$$

$$\int_Y \|p(s, x_1, y) - p(s, x_2, y)\|^2 \lambda(dy) v$$
$$\left(\int_Y \|p(s, x_1, y) - p(s, x_2, y)\|^4 \lambda(dy) \right)^{1/2} \leq K(\|x_1 - x_2\|^2)$$
$$\left(\int_Y \|p(s, x_1, y)\|^4 \lambda(dy) \right)^{1/2} \leq K(\|x_1\|^2),$$

where $K(.)$ is a concave nondecreasing function from R_+ to R_+ such that $K(0) = 0, K(\varepsilon) > 0 \ for \ \varepsilon > 0$ and

$$\int_{0+}\left(\frac{d\varepsilon}{K(\varepsilon)}\right) = +\infty$$

H2: $\forall t \in (0,b], \exists M_0 \geq 0$ so that

$$\|f(t,0)\|^2 \vee \int_Y \|h(t,0,y)\|^2 \lambda(dy) \leq M_0$$

3. Main Result. In this section the existence and uniqueness of mild solution of system (1.1) is proved by application of Borel-Cantelli Lemma (Li, 2015) and Bihari inequality (Li, Yan and C2012) to the sequence of successive approximations defined as

$$\begin{aligned}
x^0(t) &= t^{(1-\upsilon)(1-\mu)} S_{\upsilon,\mu}(t) x_0,\ 0 < t \leq b, \\
x^n(t) &= t^{(1-\upsilon)(1-\mu)} S_{\upsilon,\mu}(t)(x_0 - g(x^{n-1})) \\
&\quad + t^{(1-\upsilon)(1-\mu)} \int_0^t P_\mu(t-s) f(s, x^{n-1}(s)) ds \\
&\quad + t^{(1-\upsilon)(1-\mu)} \int_0^t P_\mu(t-s) G(s, x^{n-1}(s)) dB^H(s) \\
&\quad + t^{(1-\upsilon)(1-\mu)} \int_0^t P_\mu(t-s) \int_Y p(s, x^{n-1}(s), y) \widehat{N}(ds, dy), \\
&\quad 0 < t < b, \quad n = 1, 2, \cdots
\end{aligned} \tag{3.1}$$

THEOREM 3.1 *Whenever the hypotheses* $(H1)$–$(H2)$ *are true then system* (1.1) *has unique mild solution given that* $(4M/\Gamma(\mu)^2) t^{2(1-\upsilon)(1-\mu)}(b + 2C) < 1$ *and* $1/2 < \mu < 1$.

Proof: First the sequence $x^n(t), n \geq 1$ is shown to be bounded $\forall t\ [0,b]$. Clearly $x^0(t) \in C^{\upsilon,\mu}(J, L_2(\Omega, H))$. Now using hypotheses (H1)–(H2), Doob Martingale inequality (Li, 2015) and Burkholder-Davis-Gundy inequality (Li, 2015) we get,

$$\begin{aligned}
E\|x^n(t)\|^2 &= \frac{4M}{(\Gamma(\upsilon(1-\mu)+\mu))^2}(2\|x_0\|^2 + 2\|g(x^{n-1})\|^2) \\
&\quad + \frac{4Mt^{2(1-\upsilon)(1-\mu)}}{(\Gamma(\mu))^2} \int_0^t (t-s)^{2(\mu-1)} E\|f(s, x^{n-1}(s))\|^2 ds \\
&\quad + \frac{4Mt^{2(1-\upsilon)(1-\mu)} 2Ht^{2H-1}}{(\Gamma(\mu))^2} \int_0^t (t-s)^{2(\mu-1)} E\|G(s, x^{n-1}(s))\|^2 ds
\end{aligned}$$

$$x \qquad + \frac{4Mt^{2(1-v)(1-\mu)}C}{(\Gamma(\mu))^2}\left[\int_0^t (t-s)^{2(\mu-1)}\int_Y E\|p(s,x^{n-1}(s),y)\|^2\lambda(dy)ds\right.$$
$$\left. + \int_0^t (t-s)^{2(\mu-1)}\left(\int_Y E\|p(s,x^{n-1}(s),y)\|^4\lambda(dy)\right)^{1/2} ds\right],\ 0<t<b,$$
$$n=1,2,\cdots \tag{3.2}$$

Then

$$E\|x^n(s)\|^2 \le \frac{M_3}{1-\frac{8M\widehat{M}}{\Gamma(v(1-\mu)+\mu)^2}} + \frac{M_4}{1-\frac{8M\widehat{M}}{\Gamma(v(1-\mu)+\mu)^2}}\int_0^t (t-s)^{2(\mu-1)}E\|x^n(s)\|^2 ds, \tag{3.3}$$

where M_3 and M_4 are appropriate constants. Then by Gronwall's inequality

$$E\|x^n(s)\|^2 \le \frac{M_3}{1-\frac{8M\widehat{M}}{\Gamma(v(1-\mu)+\mu)^2}}\exp\left(\frac{M_4\left(\frac{b^{2\mu-1}}{2\mu-1}\right)}{1-\frac{8M\widehat{M}}{\Gamma(v(1-\mu)+\mu)^2}}\right) \le \infty. \tag{3.4}$$

This proves that the sequence $x^n(t),\ n\ge 1$ is bounded in $C^{v,\mu}(J,L_2(\Omega,H))$.

Next it is verified that the sequence $x^n(t), n\ge 1$ is a Cauchy sequence as follows, $\forall\ m\ge n\ge 0$,

$$\sup_{t\in[0,b]} E\|x^m(t)-x^n(t)\|^2 \le \sum_{r=n}^{\infty} C_1\frac{b^{2(\mu-1)}}{2(\mu-1)+1}M_5^r\frac{b^r}{r!}\to 0,$$
$$\text{as}\ \ n\to\infty, \tag{3.5}$$

where C_1 is an appropriate constant.

Then by Borel-Cantelli Lemma $x^n(t)\to x(t)$ as $n\to\infty$. Then using Bihari inequality (Liu, Yan and Cang, 2012) it is proved that

$$\sup_{t\in[0,b]} E\|x_1(t)-x_2(t)\|^2 = 0,$$

for any two solutions x_1 and x_2. Therefore uniqueness of mild solution is also established.

4. Optimal Control. Let us consider reflexive Banach space Z in which controls u take values. Let $2^Z - \{\Phi\}$ denote nonempty convex and closed subsets of Z. The multivalued function $v : J \to 2^Z - \{\Phi\}$ and $v(.) \subset \eta$ where η is a bounded set of Z. The admissible control set $U_{ad} = \{u(.) \in L_2(\eta)|u(t) \in v(t), a.e\}$. Then $U_{ad} \neq \Phi$ and $U_{ad} \subset L_2([0,b], Z)$ is bounded closed and convex. The fractional stochastic control problem is formulated as

$$\begin{aligned} D^{\upsilon,\mu}x(t) &= Ax(t) + f(t,x(t)) + Bu(t) + G(t,x(t))\frac{dB^H(t)}{dt} \\ &\quad + \int_Y p(t,x(t),y)\widehat{N}(dt,dy), \quad t \in (0,b] \\ &\quad I_0^{(1-\upsilon)(1-\mu)}x(0) + g(x) = x_0 \end{aligned} \tag{4.1}$$

By using definition of mild solution of (1.1), for all $u \in U_{ad}$, the mild solution of the control problem is defined as

$$\begin{aligned} x(t) &= S_{\upsilon,\mu}(t)(x_0 - g(x)) + \int_0^t P_\mu(t-s)[f(s,x(s)) + Bu(s)ds] \\ &\quad + \int_0^t P_\mu(t-s)G(s,x(s))dB^H(s) \\ &\quad + \int_0^t P_\mu(t-s)\int_Y p(s,x(s),y)\widehat{N}(ds,dy), \quad \forall t \in (0,b] \end{aligned} \tag{4.2}$$

THEOREM 4.1 (Mishura, 2008) *Under the hypotheses* $(H1)$–$(H3)$ *and for each* $u \in U_{ad}$, *the system* (4.1) *is mildly solvable on* $[0,b]$ *and the solution is unique*

The problem is to minimize the performance index defined by

$$J(x,u) = \int_0^b L(t, x^u(t), u(t))dt$$

among all admissible state control pairs of system (4.1). Goal is to find an admissible control pair $(x^0, u^0) \in C([0,b], L_2(\Omega, H)) \times U_{ad}$ so that

$$J(x^0, u^0) < J(x,u), \ \forall u \in U_{ad}$$

$(H4)$ the cost functional $L : J \times H \times Z \to R \cup \infty$ is measurable , lower semicontinuous on $H \times Z$, convex on Z for all $x \in H$ and $\exists$ constants $k \geq 0, d \geq 0, j \geq 0$

$$L(t, x(t), u(t)) \geq k(t) + d(\|x\|) + j\|u\|_Z^2.$$

THEOREM 4.2 (Mishura, 2008) *Under the hypotheses* $(H1)$–$(H4)$ *and for each* $u \in U_{ad}$, *the system* (4.1) *permits atleast one optimal control pair.*

Proof: Aim is to find an admissible control pair that minimizes the value of $J(x,u) = \int_0^b L(t, x^u(t), u(t))dt$. If $inf_{(x,u)\in H\times U_{ad}} J(x,u) = \infty$ then there is nothing to prove. Assume that $inf_{(x,u)\in H\times U_{ad}} J(x,u) = \epsilon < \infty$, then there exists a minimizing sequence feasible pair $\{(x_n, u_n)\}$ so that

$$J(x_n, u_n) \to \epsilon \ \text{ as } \ n \to \infty.$$

As $u_n \in U_{ad}$, $\{u_n\} \subset L_2(J, Z)$ is bounded. So there exists $\overline{u}$ such that $u_n \rightharpoonup u$ weakly in $L_2(J, Z)$. Let x_n and $\overline{x}$ be corresponding mild solutions to u_n and $\overline{u}$. Using closedness and convexity of U_{ad} and Mazur's Theorem and Lebesgue Dominated convergence theorem it is shown that

$$E\|x_n(t) - \overline{x}(t)\|^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore by Balder's theorem

$$J(\overline{x}, \overline{u}) \leq \lim_{n\to\infty} J(x_n, u_n) = m.$$

This shows J attains minimum at $\overline{u} \in U_{ad}$

5. Conclusion. The existence and uniqueness of mild solution of a Hilfer fractional differential system with Poisson jump and fractional Brownian motion have been established. The results are obtained by the method of successive approximation, stochastic analysis, compact semigroup theory and fractional calculus. Eventually, the existence of optimal control pair for the Hilfer fractional control system is discussed by taking general mild assumptions of cost functional.

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